The Cookie Problem

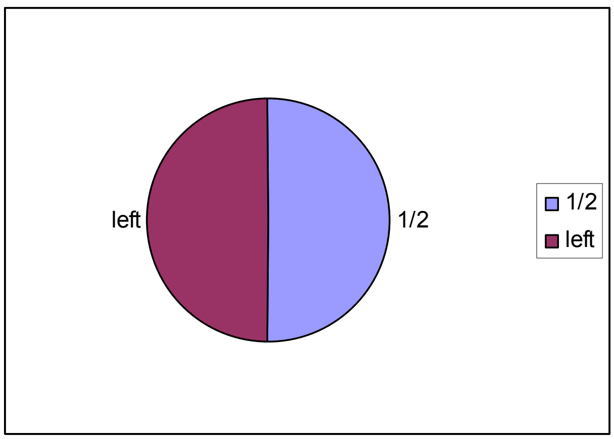
Part I

Section 1

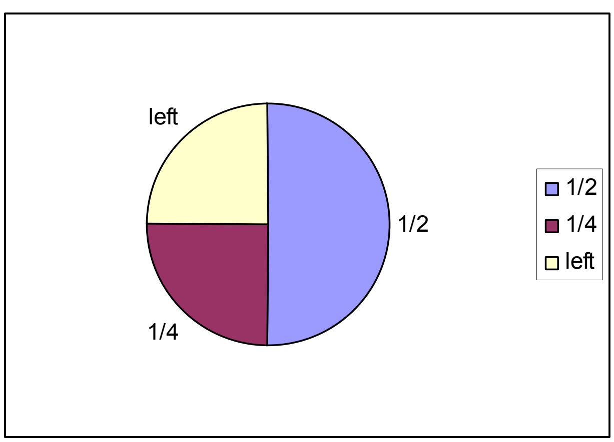
"The cookie monster sneaks into the kitchen and eats half a cookie; on the second day he comes in and eats half of what remains of the cookie from the first day; on the third day he comes in and eats half of what remains from the second day. If the cookie monster continues this process for seven days, how much of the cookie has he eaten? How much is left?"

Each day, the amount of cookie remaining is half of that from the previous day. The amount we start with is 1.

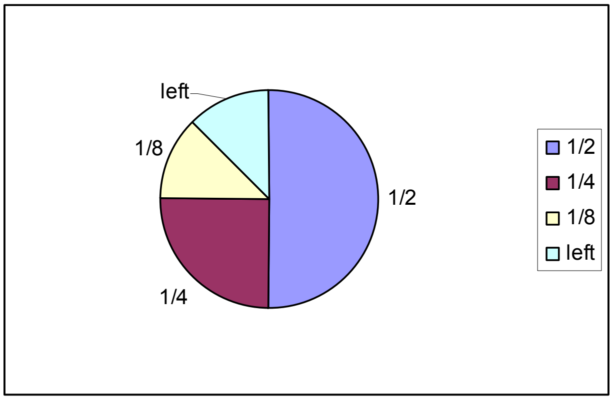
1. On the first day, the cookie monster comes in and eats half the cookie. $\frac{1}{2}$ is remaining. $1-\frac{1}{2}=\frac{1}{2}$ was eaten.



2. On the second day, the cookie monster eats half of what remains. $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^2=\frac{1}{4}$ is remaining. $1-\frac{1}{4}=\frac{3}{4}$ was eaten.



3. On the third day, the cookie monster eats half of what remains from the second day.
$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^3=\frac{1}{8}$ is remaining. $1-\frac{1}{8}=\frac{7}{8}$ was eaten.



4. …

The pattern is this:

For the n^{th} day, $\left(\frac{1}{2}\right)^n$ of the cookie is remaining. The amount not remaining must have been eaten, so that equates to $1-\left(\frac{1}{2}\right)^n$.

Another way to write the amount eaten, $1-\left(\frac{1}{2}\right)^n$, is using an infinite series.

On the first day, $\frac{1}{2}$ of the cookie was eaten. On the second day, $\left(\frac{1}{2}\right)^2$ of the cookie was eaten. The total amount of cookie eaten by the second day is $\frac{1}{2}+\left(\frac{1}{2}\right)^2$. On the third day, this becomes $\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3$, and so on. Thus, the infinite series is:

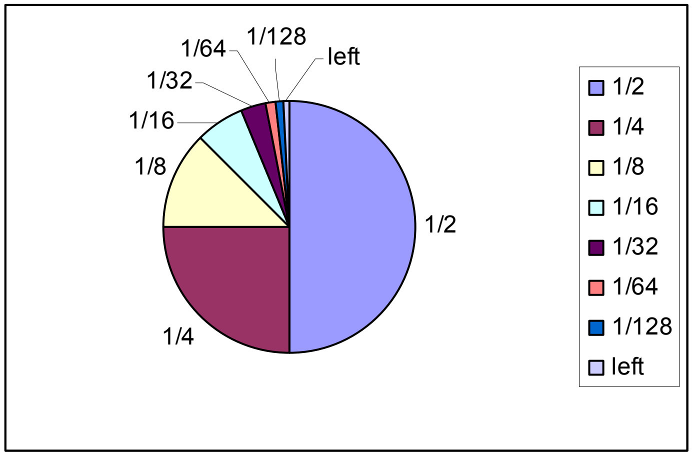
$$\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+...$$

or

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$$

The nth term is $\left(\frac{1}{2}\right)^n$ and the sum of n terms is $\sum_{k=1}^{n}\left(\frac{1}{2}\right)^k$.

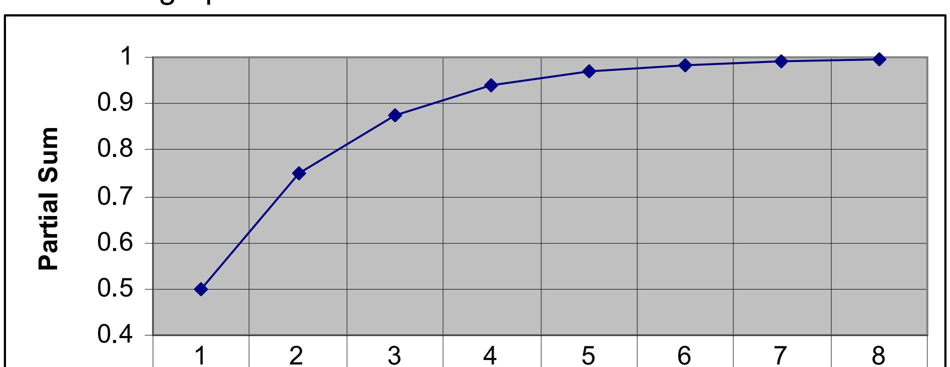
After seven days, this is what happens:



As you can see, almost the entire cookie has been eaten. This shows that the infinite series, $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k}$ approaches 1.

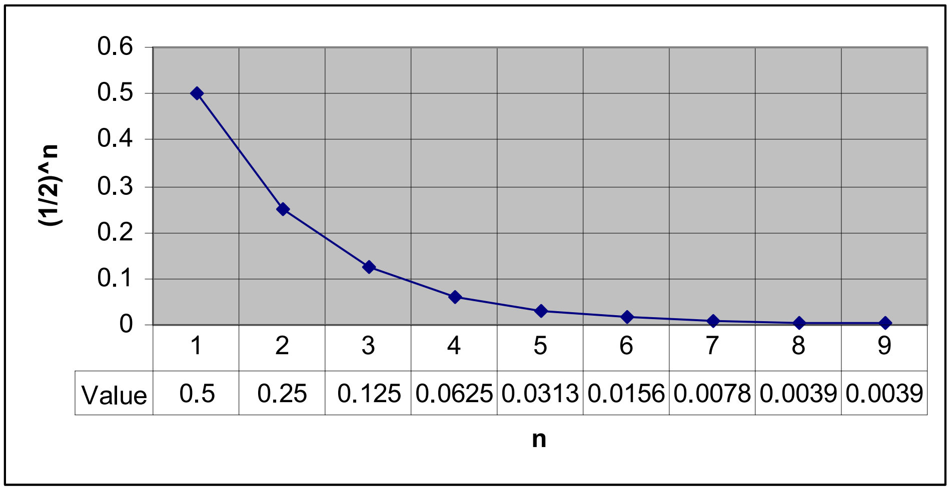
The expression $\left(\frac{1}{2}\right)^{n}$ becomes infinitely small, when n is infinite, so it approaches 0. As such, the value $1-\left(\frac{1}{2}\right)^{n}$ approaches 1.

Here is the graph of the infinite series.



Number of Terms	1	2	3	4	5	6	7	8
Value	0.5	0.75	0.875	0.9375	0.9688	0.9844	0.9922	0.9961

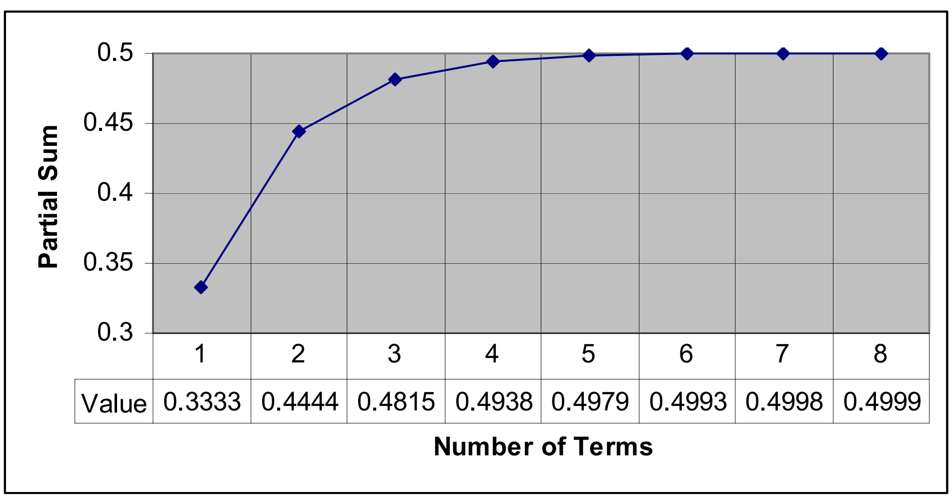
This is the graph of $y = \left(\frac{1}{2}\right)^n$.



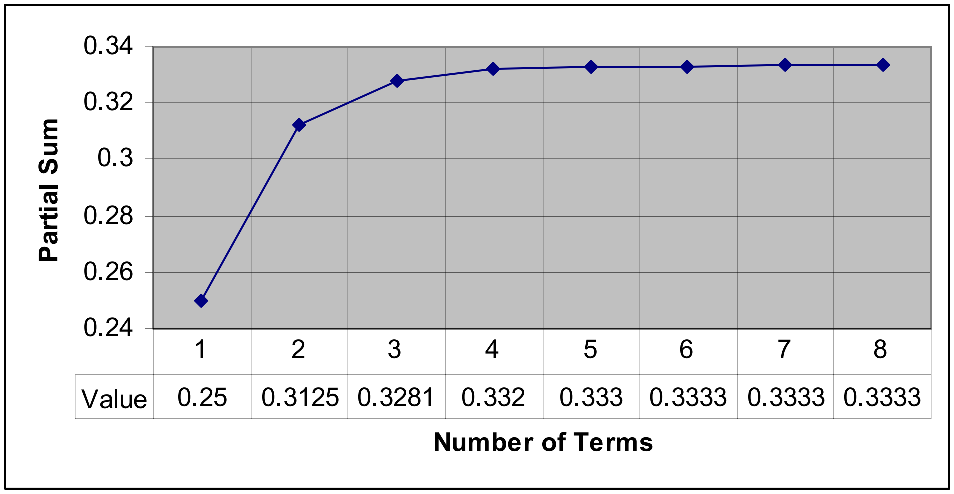
Section 2

“Pick two unit fractions and write out the sum of the infinite series. For example, you might choose 1/3+1/9+1/27+….”

This is the infinite series $\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4 + \ldots$

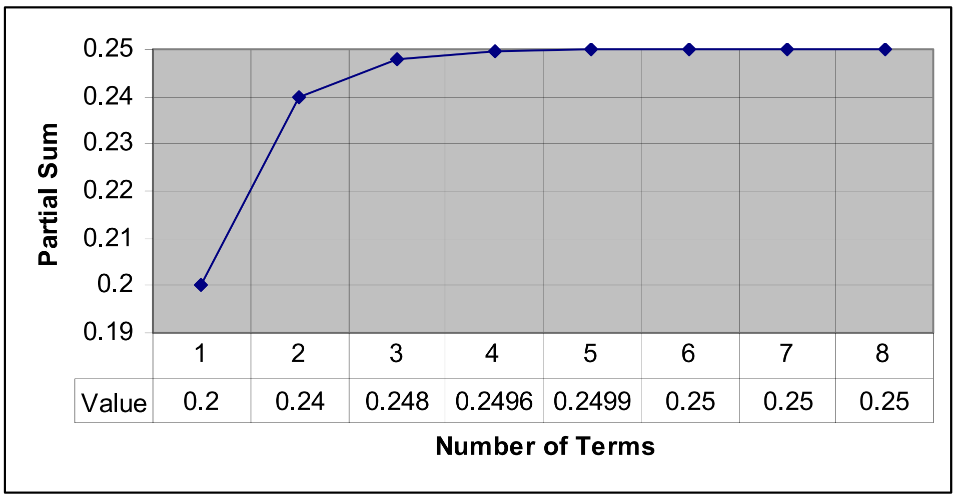


This is the infinite series $\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\ldots$



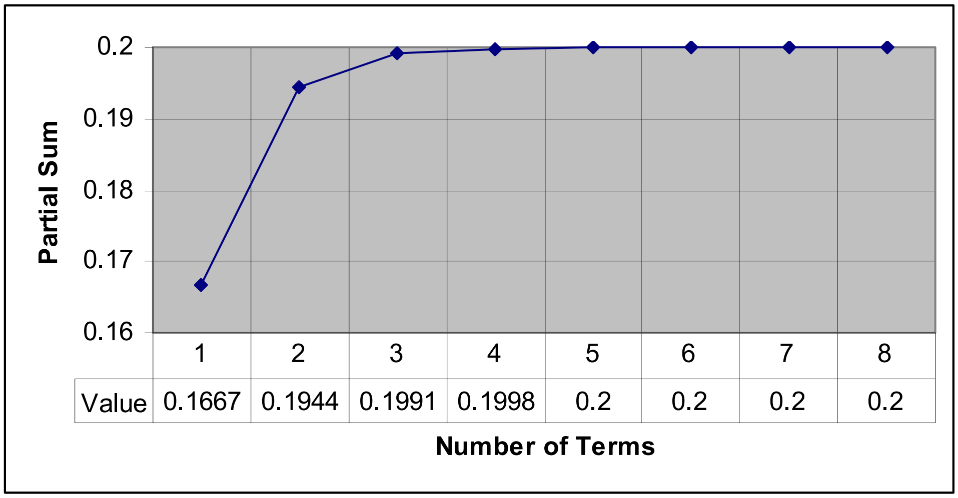
Number of Terms	1	2	3	4	5	6	7	8
Value	0.25	0.3125	0.3281	0.332	0.333	0.3333	0.3333	0.3333

This is the infinite series $\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\left(\frac{1}{5}\right)^4+\ldots$



Number of Terms	1	2	3	4	5	6	7	8
Value	0.2	0.24	0.248	0.2496	0.2499	0.25	0.25	0.25

This is the infinite series $\left(\frac{1}{6}\right)+\left(\frac{1}{6}\right)^2+\left(\frac{1}{6}\right)^3+\left(\frac{1}{6}\right)^4+\ldots$



Notice that in these examples, the infinite series using $\frac{1}{3}$ approaches $\frac{1}{2}$, the infinite series for $\frac{1}{4}$ approaches $\frac{1}{3}$, and the infinite series for $\frac{1}{5}$ approaches $\frac{1}{4}$. It appears that the infinite series $\left(\frac{1}{n}\right)+\left(\frac{1}{n}\right)^2+\left(\frac{1}{n}\right)^3+\left(\frac{1}{n}\right)^4+\ldots$ will approach $\frac{1}{n-1}$.

Here is the mathematical proof for this hypothesis. We set a variable, z, to represent the infinite series.

$$z=\sum_{k=1}^{\infty}\left(\frac{1}{n}\right)^k$$

$$z=\left(\frac{1}{n}\right)+\left(\frac{1}{n}\right)^2+\left(\frac{1}{n}\right)^3+\left(\frac{1}{n}\right)^4+\ldots$$

$$z=\left(\frac{1}{n}\right)\left[1+\frac{1}{n}+\left[\frac{1}{n}\right]^2+\left[\frac{1}{n}\right]^3+\left[\frac{1}{n}\right]^4+\ldots\right]$$

$$z=\left(\frac{1}{n}\right)\left[1+\left(\frac{1}{n}+\left[\frac{1}{n}\right]^2+\left[\frac{1}{n}\right]^3+\left[\frac{1}{n}\right]^4+\ldots\right)\right]$$

$$z = \left(\frac{1}{n}\right)\left[1 + \left(z\right)\right]$$

$$zn = 1 + z$$

$$zn - z = 1$$

$$z(n-1) = 1$$

$$z = \frac{1}{n-1}$$

Something interesting to note is that the larger n is, the faster the infinite series will converge to z. This is because $\left(\frac{1}{n}\right)^x$ will always be smaller when the exponent, x, is larger.

Part II

"**Problem:** Share 6 cookies among 7 people. **Restrictions:** You cannot use 6/7 or any non-reduced form of that fraction. Find a way to use the sum of an infinite series." (Obviously, the cookies must be divided equally.)

We can derive a formula to find the value in the infinite series, q, given the value we want it to converge to, y. Let's start by setting y equal to the infinite series.

$$y = \sum_{k=1}^{\infty} q^k$$

$$y = q + q^2 + q^3 + q^4 + \ldots$$

$$y = q\left[1 + q + q^2 + q^3 + q^4 + \ldots\right]$$

$$y = q\left[1 + \left(q + q^2 + q^3 + q^4 + \ldots\right)\right]$$

$$y = q\left[1 + \left(y\right)\right]$$

$$q = \frac{y}{1+y}$$

From here it's pretty easy to determine the infinite series that converges with $\frac{6}{7}$, which, ultimately, is the amount of cookie we want to give each person. Plug in $\frac{6}{7}$ for y, and this is what we get:

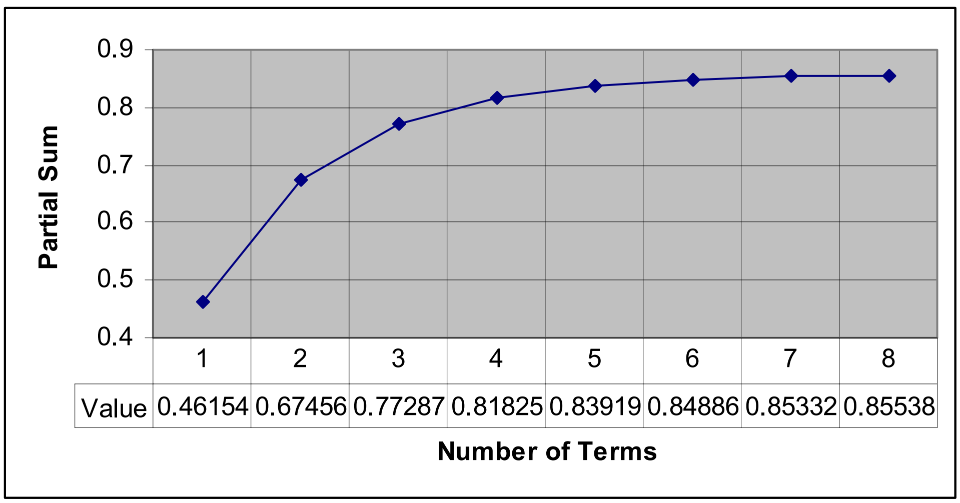
$$q = \frac{\frac{6}{7}}{1+\frac{6}{7}} = \frac{\frac{6}{7}}{\frac{13}{7}} = \frac{6}{13}$$

Now that we have determined q, we plug this back into the infinite series.

$$\left(\frac{6}{13}\right) + \left(\frac{6}{13}\right)^2 + \left(\frac{6}{13}\right)^3 + \left(\frac{6}{13}\right)^4 + \ldots$$
$$= \sum_{k=1}^{\infty} \left(\frac{6}{13}\right)^k$$

And there's our answer!

Here are the first 8 terms of the series, graphed.



	1	2	3	4	5	6	7	8
Value	0.46154	0.67456	0.77287	0.81825	0.83919	0.84886	0.85332	0.85538

As you can see, the sum is approaching $\frac{6}{7}$, or about 0.85714.

The nth term can be found by:

$$\left(\frac{6}{13}\right)^n$$

To find the sum of the first n terms of this series, evaluate the sum:

$$\sum_{k=1}^{n} \left(\frac{6}{13}\right)^k$$

or

$$\left(\frac{6}{13}\right) + \left(\frac{6}{13}\right)^2 + \left(\frac{6}{13}\right)^3 + \ldots + \left(\frac{6}{13}\right)^n$$

To actually split the cookies amongst the people, follow this procedure:

1. Cut each cookie into 13 parts.
2. Distribute 6 of these parts to each person.
3. Cut each of the remaining parts into 13 new parts.
4. Distribute 36 of these parts to each person.
5. Cut each of *these* remaining parts into 13 new parts.
6. Distribute 216 of these parts to each person.
7. …and so on and so forth. The number of parts to distribute is always 6 times greater than from the previous iteration. (powers of 6)